DATE OF EXAM - November 27, 2009 **Solution**
SUBJECT NAME -Analysis -I - MIDTERM Exam - Semester I

1. Let y be any irrational number in $[0,1]$. Let $\frac{p_n}{q_n} \to y$, p_n, q_n are positive intergers with $gcd\{p_n, q_n\} = 1$. Show that $q_n \to \infty$.

   Solution: 1. Lets assume that q_n is a bounded sequence then there is a subsequence q_{n_j} which will converge. Since q_n are positive intergers so q_{n_j} evenetually a constant sequence say $q_{n_j} = b$ for large enough n_j. Also we have $\frac{p_{n_j}}{q_{n_j}} \to y$ as $n_j \to \infty$. Which give $\lim p_{n_j} = b\ y$. Since p_{n_j} are positive integer so we get $p_{n_j} = a$ for large enough n_j. So we get $a = b\ y$ which is contradiction to the fact y is irrational. □

2. $f : [0,1] \to [0,\infty)$ be any function assume that there exists $M \geq 0$ such that for all subsets $\{x_1, x_2, \cdots, x_k\}$ of $[0,1]$, one has
   $f(x_1) + f(x_2) + \cdots + f(x_n) \leq M$. Show that $G = \{x : f(x) \neq 0\}$ is a counatable set.

   Solution: 2. We have $G = \cup_{n=1}^{\infty}\{x : f(x) \geq \dfrac{1}{n}\}$. Set $A_n = \{x : f(x) \geq \frac{1}{n}\}$. We will prove each A_n is finite. Let A_{n_0} is a infinite set. Let $k = n[2M+2]$, chose $\{y_k\}_{k+1}^{n}$ from the set A_{n_0} then so we get $f(y_1) + f(y_2) + \cdots + f(y_k) \geq \frac{1}{n}n[2M+2] > M$. Which is contradiction to the given condition. So each A_n each finite therefore G is countable set. □

3. Let f be a bounded increasing function on $(0,1)$. Let x_0 be in $(0,1)$. Show that the left limit for f at x_0 viz $\lim\limits_{x \to x_0}\{f(x) : x < x_0\}$ exists and $\sup\limits_{x<x_0} f(x)$.

   Solution: 3. Take a increasing sequence x_n such that $x_n < x_0$ and $x_n \to x_0$, then we will have $f(x_n) \leq f(x_{n+1})$. We have a incresing sequence $\{f(x_n)\}_n$ which is bounded above by $f(x_0)$ (since f is increasing) so $\sup\limits_n f(x_n)$ exists (l.u.b property) and $f(x_n) \to \sup\limits_n f(x_n)$. Now it is easy to see that $\sup\limits_n f(x_n) = \sup\limits_{x<x_0} f(x)$. Now take any sequence $\{y_n\}$ with $y_n < x_0$ such that $y_n \to x_0$. Then every subsequence y_{n_k} has a further increasing subsequence $y_{n_{k_j}}$ such that $f(y_{n_{k_j}}) \to \sup\limits_{x<x_0} f(x)$ goes to $\sup\limits_{x<x_0} f(x)$ by the above argument. So $\lim\limits_{x\to x_0}\{f(x) : x < x_0\} = \sup\limits_{x<x_0} f(x)$. □

4. $g : [0,1] \longrightarrow \mathbb{R}$ be any continous function with $g(0) < 0 < g(1)$. Show that g assumes the value 0.

   Solution: 4. Since g is continous we have $g[0,1]$ is connected as $[0,1]$ is connected. Now $g(0), g(1) \in g[0,1]$. Now connectedness of $g[0,1]$ will give $0 \in g[0,1]$ as $g(0) < 0 < g(1)$, i.e $g(x_0) = 0$ for some $x_0 \in (0,1)$. □

5. Let $f : \mathbb{R} \to \mathbb{R}$ be any infinitely differentiable function. State and prove Taylor's theorem on the interval $[x_0, x_0 + h]$ involving $f, f', f^{(2)} \cdots f^{(n+1)}$ for any $n \geq 1$.

   Solution: 5. See 5.15 Theorem of W Rudin principles of mathematical analysis with $a = x_0$ and $b = x_0 + h$ □

6. Let $f : [a,b] \to \mathbb{R}$ is continous, differentiable and f' be continous. Show that

$$\lim_{\delta\to 0}\sup_{a\leq x\leq b}\sup_{0<|t-x|<\delta}\Big|\frac{f(t)-f(x)}{t-x} - f'(x)\Big| = 0$$

.

Solution: 6. Since f is differentiable we have $f(t)-f(x)=(t-x)f'(c),\quad a<c<b$. Since f' is continous on $[a,b]$ so its uniformly continous i.e $\lim_{\delta\to 0}\sup_{a\le u\le b}\sup_{0<|u-v|<\delta}|f'(u)-f'(v)|=0$. Now using this we have

$$\begin{aligned}\lim_{\delta\to 0}\sup_{a\le x\le b}\sup_{0<|t-x|<\delta}\Big|\frac{f(t)-f(x)}{t-x}-f'(x)\Big| &= \lim_{\delta\to 0}\sup_{a\le x\le b}\sup_{0<|c-x|<\delta}\big|f'(c)-f'(x)\big|,\quad \min\{t,x\}<c<\max\{t,x\}\\ &=0.\end{aligned}$$

□

7. Let $a_1,a_2,\cdots,a_n$ be a sequence of reals converging to 0. Let $\alpha:\{1,2,3,\cdots\}\to\{1,2,3,\cdots\}$ be any 1-1 and onto map. Put $b_j=a_{\alpha j}$. show that $b_j\to 0$.

   Solution:7 Since $a_n\to 0$ we have $|a_n|<\epsilon\quad n\ge M$. We can alway chose $N>M$ such that $\{1,2,\cdots,M\}\subset\{\alpha(1),\alpha(2),\cdots,\alpha(N)\}$, $(N>\max_{1\le j\le M}\alpha^{-1}(j))$.
   Now we have $\{a_{\alpha(j)}:j\ge N\}\subset\{a_j:j\ge M\}$ this will give $|b_j|=|a_{\alpha(j)}|<\epsilon\quad j\ge N$. □

8. Let a and b be real numbers. If the series $(a+b)+(a+2b)+(a+3b)+\cdots$ is convergent, then show that $b=0$ and $a=0$.

   Solution:8 Since $\sum_n(a+nb)<\infty$ we have $\lim_{n\to\infty}(a+nb)=0$. If we assume $b\ne 0$ then we get $\infty=\lim_{n\to\infty}n=-\dfrac{a}{b}<\infty$. So we get $b=0$. Once we get $b=0$ we have $a=0$. □

9. A,B be bounded subsets of $[0,\infty)$. Let $C=\{ab:a\in A,b\in B\}$. Let $x=supA$, $y=supB$, $z=supC$. Note that x need not be in A and y need not be in B. Show that $z=xy$.

   Solution:9 It is easy to see that $supC\le ab$. We can find sequences $\{a_n\}\subset A$ and $\{b_n\}\subset B$ such that $a_n\to x$ and $b_n\to y$ this will imply $a_nb_n\to ab$ $\big(a_nb_n-ab=a_n(b_n-b)+(a_n-a)b\big)$ this together with $a_nb_n\in C$ will give $z=xy$. □

10. (a) Let $a_1,a_2,$ be a sequence with $a_j\ge 0$. Let $\sum_1^\infty a_j$ be convergent, let $n_1<n_2<\cdots$ be increasing sequence of natural numbers let $b_j=a_{n_j}$. Show that $\sum_j b_j$ is convergent.

    (b) Give an example of a real sequence $x_1,x_2,\cdots$ and a subsequence $x_{n_1},x_{n_2},\cdots$ such that $\sum x_j$ is convergent but $\sum x_{n_j}$ is not convergent. Prove your claim.

    Solution: 10.(a) Since $|\sum_{j=M}^\infty a_j|<\epsilon$. We can find K such that $\{b_j=a_{n_j}:j\ge K\}\subset\{a_j:j\ge M\}\subset$. So we have $|\sum_{j=K}^\infty b_j|\le|\sum_{j=M}^\infty a_j|<\epsilon$.

    (b) Set $x_n=(-1)^{n-1}\frac{1}{n}$ and $x_{n_j}=x_{2j}$. Then $\sum x_{n_j}=-\sum\frac{1}{2j}$ is divergent. Now define $S_{2N}=\sum_{n=1}^{2N}(-1)^{n-1}\dfrac{1}{n}$. Then

$$S_{2N}=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{1}{2N-1}-\frac{1}{2N}=1-(\frac{1}{2}-\frac{1}{3})-(\frac{1}{4}-\frac{1}{5})-\cdots-(\frac{1}{2N-2}-\frac{1}{2N-1})-\frac{1}{2N}<1.$$

Now $S_2 = 1 - \frac{1}{2}$, $S_4 = 1 - \frac{1}{2} + (\frac{1}{3} - \frac{1}{4}) > S_2$, $S_6 = 1 - \frac{1}{2} + (\frac{1}{3} - \frac{1}{4}) + (\frac{1}{5} - \frac{1}{6}) > S_4$. In this way we can see that $\{S_{2N}\}_N$ is incresing sequence but bounded above by 1. So $\lim S_{2N}$ exists. Now $S_{2N+1} = S_{2N} + \frac{1}{2N+1}$ which will give $\lim S_{2N} = \lim S_{2N+1}$, so S_N is convergent. $\square$

11. Let $a_j > 0$ for $j = 1, 2, 3, \cdots$. Assume that $\frac{a_{j+1}}{a_j}$ and $a_j^{\frac{1}{j}}$ are also bounded sequence. Show that $\limsup_{j\to\infty} a_j^{\frac{1}{j}} \leq \limsup_{j\to\infty} \frac{a_{j+1}}{a_j}$.

    Solution:11 See 3.37 Theorem of W Rudin (principles of mathematical analysis). $\square$

12. Define $f : [0,1] \to \mathbb{R}$ by $f(x) = x^a \sin(\frac{1}{x^c})$ where $c > 0$ and $a \geq 0$ for $x > 0$, $f(0) = 0$.
    (i) If f is continous at 0, show that $a > 0$.
    (ii) If f is differentiable at 0, show that $a > 1$.

    Solution:12 (i) Let assume $a \leq 0$. Then take $x_n = \left(\frac{2}{(2n+1)\pi}\right)^{\frac{1}{c}} \leq 1$. Now we have

    $$f(x_n) = \left(\frac{2}{(2n+1)\pi}\right)^{\frac{a}{c}} \sin\frac{(2n+1)\pi}{2} \to \infty \ (or \ \ 1) \ \ as \ \ n \to \infty, \ \ as \ \ a < 0 \ (or \ a = 0).$$

    So we get if $a \leq 0$, f is not continous at 0. On the other hand if $a > 0$ we get

    $$\left|x^a \sin\frac{1}{x^c}\right| \leq |x|^a < \epsilon \ \ as \ \ |x| < \epsilon^{\frac{1}{a}}.$$

    (ii) If f is differentiable at 0 then following limit has to exist.

    $$\lim_{x\to 0} \frac{x^a \sin\frac{1}{x^c}}{x} = \lim_{x\to 0} x^{a-1} \sin\frac{1}{x^c}.$$

    The above limit will exist if $a > 1$ so we get f is differentiable at 0 then $a > 1$. $\square$

13. Let $f : [0,\infty) \to \mathbb{R}$ be continuous and differentiable on $(0,\infty)$. Further let $f(0) = 0$ and f' is increasing on $(0,\infty)$. Define $g(x) = \frac{f(x)}{x}$ for $x > 0$. Show that g is increasing on $(0,\infty)$.

    Solution: 13 Now for $x > 0$ we have $g'(x) = \frac{1}{x}\left[f'(x) - \frac{f(x)}{x}\right]$. MVT will give $f(x) = xf'(c) \ \ 0 < c < x$. So $g'(x) = \frac{1}{x}[f'(x) - f'(c)] > 0$ as $0 < c < x$ and f' is increasing for $x > 0$. $g'(x) > 0$ for $x > 0$ imply g is increasing on $(0,\infty)$. $\square$

14. Let $a_{ij} : \mathbb{R} \to \mathbb{R}$ be C^∞ functions for $i, \ j = 1, 2, 3$ Let $\alpha(x) = det\big[\big(a_{ij}(x)\big)\big]$. show that the derivative of α can be written as a sum of three determinants involving a_{ij} and its derivatives. Be as explicit as possible.

    Solution: 14 A calculation will give the following

    $$\alpha'(x) = \begin{vmatrix} a'_{11}(x) & a'_{12}(x) & a'_{13}(x) \\ a_{21}(x) & a_{22}(x) & a_{23}(x) \\ a_{31}(x) & a_{32}(x) & a_{33}(x) \end{vmatrix} + \begin{vmatrix} a_{11}(x) & a_{12}(x) & a_{13}(x) \\ a'_{21}(x) & a'_{22}(x) & a'_{23}(x) \\ a_{31}(x) & a_{32}(x) & a_{33}(x) \end{vmatrix} + \begin{vmatrix} a_{11}(x) & a_{12}(x) & a_{13}(x) \\ a_{21}(x) & a_{22}(x) & a_{23}(x) \\ a'_{31}(x) & a'_{32}(x) & a'_{33}(x) \end{vmatrix}.$$

    $\square$

15. Let $f : \mathbb{R} \to \mathbb{R}$ is a C^2 function i.e $f,\ f',\ f''$ all are continous. Assume that $M_0 = \sup_x |f(x)|$ and $M_2 = \sup_x |f''(x)|$ are both finite. Put $M_1 = \sup_x |f'(x)|$. Show that $M_1^2 \leq M_0\ M_2$.

Solution:15 Taylor's theorem will give

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2} f''(\xi), \quad x < \xi < x+h.$$

From above we get

$$f'(x) = \frac{1}{h}[f(x+h) - f(x)] - \frac{h^2}{2} f''(\xi) \ \Rightarrow\ |f'(x)| \leq \frac{2}{h} M_0 + \frac{h}{2} M_2$$

If we put $h = 2\sqrt{\frac{M_0}{M_2}}$ in the above we get $M_1 \leq 2\sqrt{M_0 M_2}$ i.e $M_1^2 \leq M_0\ M_2$. □

16. Let $a_1, a_2, \cdots$ be a sequence of reals with $\sum a_j$ convergent. Let $n_1 < n_2 < n_3 \cdots\cdots$. Put $b_1 = a_1 + a_2 + \cdots + a_{n_1}$, $b_2 = a_{n_1+1} + a_{n_1+2} + \cdots + a_{n_2}$, $b_3 = a_{n_2+1} + a_{n_2+2} + \cdots + a_{n_3}$ and so on. Show that the series $b_1 + b_2 + b_3 + \cdots$ convergent and converges to $\sum a_r$.

Solution: 16 We have $\displaystyle\sum_{k=N}^{M} b_k = \sum_{j=n_{N-1+1}}^{n_M} a_j, \quad M > N$. Now the fact $N \to \infty$ imply $n_{N-1} + 1,\ n_M \to \infty$ give the convergence of $\sum b_j$. Now for any $N \in \mathbb{N}$ we have $n_k \leq N < n_{k+1}$ for some n_k.

$$\sum_{j=1}^{N} b_j - \sum_{j=1}^{N} a_j = \sum_{j=n_k+m+1}^{n_M} a_j, \quad N = n_k + m, \quad m \geq 0.$$

Now $N \to \infty$ imply $n_k + m + 1,\ n_M \to \infty$, So we will get $\displaystyle\lim_{N\to\infty} \left| \sum_{j=1}^{N} b_j - \sum_{j=1}^{N} a_j \right| = 0$.

In above we use the fact that $\displaystyle\sum_{k=N}^{M} a_k \to 0$ as $N, M \to \infty$ ($\sum_k a_k$ is convergent). □

17. Show that any disjoint collection of bounded intervals each of positive length is finite or countable.

Solution: 17 Let $\{I_\alpha\}_{\alpha\in A}$ be a uncountable disjoint collection of bounded intervals each has positive length. Since rationals $\mathbb{Q}$ are dense in $\mathbb{R}$, so from each I_α we can chose a rational number $x_\alpha\ \ (\in I_\alpha \cap \mathbb{Q})$. Since $I_\alpha \cap I_\beta = \emptyset\ \ \alpha \neq \beta$, we get a $\{x_\alpha\}_{\alpha\in A}$ uncountable collection of rationals, which is not possible. So $\{I_\alpha\}_{\alpha\in A}$ has to be a finite or countable collection. □

18. Let $f : (a, b) \to \mathbb{R}$ be continous every where and differentiable at x_0 in (a, b). Let $x_0 < a_n < b_n$ and $b_n \to x_0$, if $\frac{b_n - x_0}{b_n - a_n}$ is a bounded sequence, show that $\frac{f(b_n) - f(a_n)}{b_n - a_n} \to f'(x_0)$.

Solution: 18 Let $b_n = x_0 + h_n$, $a_n = x_0 + u_n$, $0 < u_n < h_n$. So we have $h_n,\ u_n \to 0$ (as $b_n \to x_0$). Given that $\frac{b_n - x_0}{b_n - a_n} = \frac{h_n}{h_n - u_n}$ is bounded, so there exist a subsequence $\frac{h_{n_k}}{h_{n_k} - u_{n_k}} = \frac{b_{n_k} - x_0}{b_{n_k} - a_{n_k}} \to \displaystyle\liminf_n \frac{h_n}{h_n - u_n}$. Since $\frac{u_{n_k}}{h_{n_k} - u_{n_k}}$ is also bounded we have further subsequence $\frac{u_{n_{k_j}}}{h_{n_{k_j}} - u_{n_{k_j}}} \to \displaystyle\limsup_{n_k} \frac{u_{n_k}}{h_{n_k} - u_{n_k}}$.

$$\begin{aligned}
\liminf_n \frac{f(b_n)-f(a_n)}{b_n-a_n} &\geq \liminf_n \frac{f(x_0+h_n)}{h_n}\frac{h_n}{h_n-u_n} + \liminf_n \frac{f(x_0+u_n)}{h_n}\frac{-u_n}{h_n-u_n} \\
&= f'(x_0)\liminf_n \frac{h_n}{h_n-u_n} - f'(x_0)\limsup_n \frac{u_n}{h_n-u_n} \\
&= f'(x_0)\left[\lim_{k_j\to\infty} \frac{h_{n_{k_j}}}{h_{n_{k_j}}-u_{n_{k_j}}} - \lim_{k_j\to\infty} \frac{u_{n_{k_j}}}{h_{n_{k_j}}-u_{n_{k_j}}}\right] \\
&= f'(x_0).
\end{aligned}$$

So we get $\liminf_n \dfrac{f(b_n)-f(a_n)}{b_n-a_n} \geq f'(x_0)$.

Similar method as above will give $\limsup_n \dfrac{f(b_n)-f(a_n)}{b_n-a_n} \leq f'(x_0)$. So we get $\frac{f(b_n)-f(a_n)}{b_n-a_n} \to f'(x_0)$.